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Ying Hu* and Shanjian Tang†

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Abstract

This paper is concerned with the switching game of a one-dimensional backward stochastic differential equation (BSDE). The associated Bellman-Isaacs equation is a system of matrix-valued BSDEs living in a special unbounded convex domain with reflection on the boundary along an oblique direction. In this paper, we show the existence of an adapted solution to this system of BSDEs with oblique reflection by the penalization method, the monotone convergence, and the a priori estimates.

Key Words. switching game, backward stochastic differential equations, oblique reflection.

Abbreviated title. Switching game of BSDEs and associated obliquely reflected BSDEs

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*IRMAR, Université Rennes 1, Campus de Beaulieu, 35042 Rennes Cedex, France. Part of this work was completed when this author was visiting School of Mathematical Sciences, Fudan University, whose hospitality is greatly appreciated. *E-mail:* `Ying.Hu@univ-rennes1.fr`.

†Corresponding author. School of Mathematical Sciences, Fudan University, Shanghai 200433, China. Part of this work was completed when this author was visiting IRMAR, Université Rennes 1, whose hospitality is greatly appreciated. *E-mail:* `sjtang@fudan.edu.cn`.

1 Introduction

In this paper, we study the system of reflected BSDEs along an oblique direction arising naturally from the problem of switching game of a scalar-valued BSDE. Let us first describe precisely the switching game problem by introducing some notations and hypotheses.

Let us fix a nonnegative real number $T > 0$. First of all, $W = \{W_t\}_{t\geq 0}$ is a standard Brownian motion with values in R^d defined on some complete probability space $(\Omega, \mathcal{F}, P)$. $\{\mathcal{F}_t\}_{t\geq 0}$ is the natural filtration of the Brownian motion W augmented by the P-null sets of $\mathcal{F}$. All the measurability notions will refer to this filtration. In particular, the sigma-field of predictable subsets of $[0, T] \times \Omega$ is denoted by $\mathcal{P}$. Define $\Lambda := \{1, \cdots, m_1\}$ and $\Pi := \{1, \cdots, m_2\}$.

We denote by $S^2(R^{m_1\times m_2})$ or simply by S^2 the set of $R^{m_1\times m_2}$-valued, adapted and càdlàg processes $\{Y(t)\}_{t\in[0,T]}$ such that

$$||Y||_{S^2} := E\left[\sup_{t\in[0,T]} |Y(t)|^2\right]^{1/2} < +\infty.$$

$(S^2, ||\cdot||_{S^2})$ is then a Banach space.

We denote by $M^2((R^{m_1\times m_2})^d)$ or simply by M^2 the set of (equivalent classes of) predictable processes $\{Z(t)\}_{t\in[0,T]}$ with values in $(R^{m_1\times m_2})^d$ such that

$$||Z||_{M^2} := E\left[\int_0^T |Z(s)|^2 ds\right]^{1/2} < +\infty.$$

M^2 is then a Banach space endowed with this norm.

We define also

$$\begin{aligned} N^2(R^{m_1\times m_2}) : \quad = \quad & \{K = (K_{ij}) \in S^2 : \text{ for any } (i, j) \in \Lambda \times \Pi, K_{ij}(0) = 0, \\ & \text{and } K_{ij}(\cdot) \text{ is increasing } \}, \end{aligned}$$

which is abbreviated as N^2. $(N^2, ||\cdot||_{S^2})$ is then a Banach space.

Let ψ be a random function $\psi : [0, T] \times \Omega \times R \times R^d \times \Lambda \times \Pi \to R$ whose component $\psi(\cdot, i, j)$ is measurable with respect to $\mathcal{P} \otimes \mathcal{B}(R) \otimes \mathcal{B}(R^d)$ for each pair $(i, j) \in \Lambda \times \Pi$, and satisfies the following Lipschitz condition.

Hypothesis 1.1. *(i) The generator* $\psi(\cdot, 0, 0) := (\psi(\cdot, 0, 0, i, j))_{i\in\Lambda, j\in\Pi} \in M^2$.

(ii) There exists a constant $C > 0$ *such that, for each* $(t, y, y', z, z', i, j) \in [0, T] \times R \times R \times R^d \times R^d \times \Lambda \times \Pi$,

$$|\psi(t, y, z, i, j) - \psi(t, y', z', i, j)| \leq C(|y - y'| + |z - z'|), \quad a.s.$$

The functions k and l are defined on $\Lambda \times \Lambda$ and $\Pi \times \Pi$, respectively; their values are both positive. We make the following assumption on the functions k and l, which is standard in the literature.

Hypothesis 1.2. *(i) For $i \in \Lambda$, $k(i,i) = 0$. For $(i,i') \in \Lambda \times \Lambda$ such that $i \neq i'$, $k(i,i') > 0$.*

(ii) For $j \in \Pi$, $l(j,j) = 0$. For $(j,j') \in \Pi \times \Pi$ such that $j \neq j'$, $l(j,j') > 0$.

(iii) For any $(i,i',i'') \in \Lambda \times \Lambda \times \Lambda$ such that $i \neq i'$ and $i' \neq i''$,

$$k(i,i') + k(i',i'') > k(i,i'').$$

(iv) For any $(j,j',j'') \in \Pi \times \Pi \times \Pi$ such that $j \neq j'$ and $j' \neq j''$,

$$l(j,j') + l(j',j'') > l(j,j'').$$

Definition 1.1. *An admissible switching process for Player I (resp. II) on $[t,T]$ with initial value $a_0 \in \Lambda$ (resp. $b_0 \in \Pi$) is defined to be a pair of sequences $\{a_i, \theta_i\}_{i \geq 0}$ (resp. $\{b_i, \tau_i\}_{i \geq 0}$), such that each θ_i (resp. τ_i) is an $\mathcal{F}_{\cdot}$-stopping time with*

$$t = \theta_0 \leq \theta_1 \leq \cdots \leq T, \quad \text{a.s.}$$

$$(\textit{ resp. } t = \tau_0 \leq \tau_1 \leq \cdots \leq T, \quad \text{a.s. }),$$

each a_i (resp. b_i) is $\mathcal{F}_{\theta_i}$ (resp. $\mathcal{F}_{\tau_i}$) measurable with values in Λ (resp. Π), and there is an integer-valued random variable $N(\cdot)$ satisfying

$$\theta_N = T \quad (\text{resp. } \tau_N = T) \quad P\text{-}a.s. \quad \text{and} \quad N \in L^2(\mathcal{F}_T).$$

Denote by $\mathcal{A}^a[t,\hat{t}]$ (resp. $\mathcal{B}^b[t,\hat{t}]$) the totality of the admissible switchings for Player I (resp. II) on $[t,\hat{t}]$ with the initial value $a \in \Lambda$ (resp. $b \in \Pi$). Define the following abbreviations for $t \in [0,T]$:

$$\mathcal{A}_t^a := \mathcal{A}^a[t,T], \quad a \in \Lambda; \quad \mathcal{A}_t := \bigcup_{a \in \Lambda} \mathcal{A}_t^a$$

and

$$\mathcal{B}_t^b := \mathcal{B}^b[t,T], \quad b \in \Pi; \quad \mathcal{B}_t := \bigcup_{b \in \Pi} \mathcal{B}_t^b.$$

We shall identify $\{a_i, \theta_i\}_{i \geq 0} \in \mathcal{A}^a[t,T]$ with

$$a(s) = a_0 \chi_{\{\theta_0\}}(s) + \sum_{i=1}^{N} a_{i-1} \chi_{(\theta_{i-1},\theta_i]}(s), \quad s \in [t,T]. \tag{1.1}$$

For any $a(\cdot) \in \mathcal{A}_t$, we define the associated (cost) process $A^{a(\cdot)}$ on $[t,T]$ as follows:

$$A^{a(\cdot)}(s) = \sum_{j=1}^{N-1} k(a_{j-1}, a_j) \chi_{[\theta_j,T]}(s), \quad s \in [t,T]. \tag{1.2}$$

Obviously, $A^{a(\cdot)}(\cdot)$ is a càdlàg process. In an identical way, we define $\mathcal{B}^{b(\cdot)}$ on $[t,T]$ for $b(\cdot) \in \mathcal{B}_t$.

Definition 1.2. *For $t \in [0,T]$ and $a \in \Lambda$(resp. $b \in \Pi$), an admissible strategy $\alpha^{a,t}$ (resp. $\beta^{b,t}$) with the initial value $a \in \Lambda$ (resp. $b \in \Pi$) for player I (resp. II) on $[t,T]$ is a mapping $\alpha^{a,t} : \cup_{b\in B}\mathcal{B}^b[t,T] \to \mathcal{A}^a[t,T]$ (resp. $\beta^{b,t} : \cup_{a\in A}\mathcal{A}^a[t,T] \to \mathcal{B}^b[t,T]$) such that*

$$b(s) = \widehat{b}(s) \quad (\text{resp. } a(s) = \widehat{a}(s)) \quad a.s.\ \forall s \in [t,\hat{t}],$$

implies

$$\alpha^{a,t}[b(\cdot)](s) = \alpha^{a,t}[\widehat{b}(\cdot)](s) \quad (\text{resp. } \beta^{b,t}[a(\cdot)](s) = \beta^{b,t}[\widehat{a}(\cdot)](s))$$

for $s \in [t,\hat{t}]$.

We denote by Γ_t^a (resp. Δ_t^b) all admissible strategies with the initial value $a \in \Lambda$ (resp. $b \in \Pi$) for player I (resp. II) on $[t,T]$. We adopt the convention that

$$\mathcal{A}_T^a = \{a\}, \quad \Gamma_T^a = \{a\}$$

and

$$\mathcal{B}_T^b = \{b\}, \quad \Delta_T^b = \{b\}.$$

Let ξ be an $R^{m_1\times m_2}$-valued $\mathcal{F}_T$-measurable random variable. Now we are in position to introduce the switched BSDEs for both players. For $t \in [0,T]$, $a(\cdot) \in \mathcal{A}_t$ and $b(\cdot) \in \mathcal{B}_t$, consider the following BSDE:

$$\begin{aligned} U(s) &= \xi_{a(T)b(T)} + \left(A^{a(\cdot)}(T) - A^{a(\cdot)}(s)\right) - \left(B^{b(\cdot)}(T) - B^{(\cdot)}(s)\right) \\ &\quad + \int_s^T \psi(r,U(r),V(r),a(r),b(r))\,dr - \int_s^T V(r)\,dW(r), \quad s \in [t,T]. \end{aligned} \tag{1.3}$$

This is a (slightly) generalized BSDE: it is equivalent to the following standard BSDE:

$$\begin{aligned} \bar{U}(s) &= \xi_{a(T)b(T)} + A^{a(\cdot)}(T) - B^{b(\cdot)}(T) \\ &\quad + \int_s^T \psi(r,\bar{U}(r) - A^{a(\cdot)}(r) + B^{b(\cdot)}(r), \bar{V}(r), a(r))\,dr \\ &\quad - \int_s^T \bar{V}(r)\,dW(r), \quad s \in [t,T] \end{aligned} \tag{1.4}$$

via the simple change of variable:

$$\bar{U}(s) = U(s) + A^{a(\cdot)}(s) - B^{b(\cdot)}(s), \quad \bar{V}(s) = V(s).$$

Hence, for each pair $(a(\cdot),b(\cdot)) \in \mathcal{A}_t \times \mathcal{B}_t$, BSDE (1.3) has a unique solution in $S^2 \times M^2$, which will be denoted by $\left(U^{a(\cdot),b(\cdot)}, V^{a(\cdot),b(\cdot)}\right)$. We note that U is only a càdlàg process.

The upper and lower switching game problems with the initial scheme $(i,j) \in \Lambda\times\Pi$ are defined as follows:

$$\operatorname*{ess\,sup}_{\beta\in\Delta_t^j} \operatorname*{ess\,inf}_{a(\cdot)\in\mathcal{A}_t^i} U^{a(\cdot),\beta(a(\cdot))}(t)$$

and

$$\operatorname*{ess\,inf}_{\alpha\in\Gamma_t^i} \operatorname*{ess\,sup}_{b(\cdot)\in\mathcal{B}_t^j} U^{\alpha(b(\cdot)),b(\cdot)}(t),$$

respectively. If

$$\mathcal{Y}^{ij}(t) := \operatorname*{ess\ sup}_{\beta\in\Delta_t^j} \operatorname*{ess\ inf}_{a(\cdot)\in\mathcal{A}_t^i} U^{a(\cdot),\beta(a(\cdot))}(t) = \operatorname*{ess\ inf}_{\alpha\in\Gamma_t^i} \operatorname*{ess\ sup}_{b(\cdot)\in\mathcal{B}_t^j} U^{\alpha(b(\cdot)),b(\cdot)}(t)$$

for some $(i,j)\in\Lambda\times\Pi$, we say that the switching game with the initial scheme $(i,j)\in\Lambda\times\Pi$ has a value $\mathcal{Y}^{ij}(t)$.

The above switching game (see, e.g. [17]) is associated to the following Bellman-Isaacs equation, which is a new type of reflected backward stochastic differential equation (RBSDE for short) with oblique reflection: for $(i,j)\in\Lambda\times\Pi$ and $t\in[0,T]$,

$$\begin{cases} Y_{ij}(t) &= \xi_{ij} + \int_t^T \psi(s, Y_{ij}(s), Z_{ij}(s), i, j)\, ds \\ & \quad - \int_t^T dK_{ij}(s) + \int_t^T dL_{ij}(s) - \int_t^T Z_{ij}(s)\, dW(s), \\ Y_{ij}(t) &\le \min_{i'\neq i}\{Y_{i'j}(t) + k(i,i')\}, \\ Y_{ij}(t) &\ge \max_{j'\neq j}\{Y_{ij'}(t) - l(j,j')\}, \\ & \int_0^T \left(Y_{ij}(s) - \min_{i'\neq i}\{Y_{i'j}(s) + k(i,i')\}\right) dK_{ij}(s) = 0, \\ & \int_0^T \left(Y_{ij}(s) - \max_{j'\neq j}\{Y_{ij'}(s) - l(j,j')\}\right) dL_{ij}(s) = 0. \end{cases} \tag{1.5}$$

Here, the unknowns are the processes $\{Y(t)\}_{t\in[0,T]}$, $\{Z(t)\}_{t\in[0,T]}$, $\{K(t)\}_{t\in[0,T]}$, and $\{L(t)\}_{t\in[0,T]}$, which are required to be adapted with respect to the natural completed filtration of the Brownian motion W. Moreover, K and L are componentwisely increasing processes. The last two relations in (1.5) are called the upper and lower minimal boundary conditions.

One-dimensional RBSDEs were first studied by El Karoui et al. [7] in the case of one obstacle, and then by Cvitanic and Karatzas [4] in the case of two obstacles. In both papers, it is recognized that one-dimensional reflected BSDEs, with one obstacle and with two obstacles, are generalizations of optimal stopping and Dynkin games, respectively. Nowadays, the literature on one-dimensional reflected BSDEs is very rich. The reader is referred to Peng and Xu [14] and Buckdahn and Li [2], among others, for the one-dimensional reflected BSDEs with two obstacles.

Multi-dimensional RBSDEs were studied by Gegout-Petit and Pardoux [8], but their BSDE is reflected on the boundary of a convex domain along the inward normal direction, and their method depends heavily on the properties of this inward normal reflection (see (1)-(3) in [8]). We note that in a very special case (e.g., ψ is independent of z), Ramasubramanian [16] studied a BSDE in an orthant with oblique reflection. Multi-dimensional BSDEs reflected along an oblique direction rather than a normal direction, still remains to be open in general, even in a convex domain, let alone in a nonconvex domain. Note that there are some papers dealing with SDEs with oblique reflection (see, e.g. [12, 5]).

In our previous work [11], we studied the optimal switching problem for one-dimensional BSDEs, and the associated following type of obliquely reflected multi-dimensional BSDEs: for $i \in \Lambda$,

$$\left\{\begin{array}{rcl} Y_i(t) & = & \xi_i + \displaystyle\int_t^T \psi(s, Y_i(s), Z_i(s), i)\, ds \\ & & - \displaystyle\int_t^T dK_i(s) - \int_t^T Z_i(s)\, dW(s), \\ Y_i(t) & \le & \displaystyle\min_{i' \neq i}\{Y_{i'}(t) + k(i,i')\}, \\ & & \displaystyle\int_0^T \left(Y_i(s) - \min_{i' \neq i}\{Y_{i'}(s) + k(i,i')\}\right) dK_i(s) = 0. \end{array}\right. \tag{1.6}$$

It should be added that a less general form of RBSDE (1.6) (where the generator ψ does not depend on (y,z)) is suggested by [3]. But they did not discuss the existence and uniqueness of solution, which is considered to be difficult. See Remark 3 in [3].

Recently, Tang, Zhong and Koo [19] discussed the mixed switching and stopping problem for one-dimensional BSDEs, and obtained the existence and uniqueness result for the associated following type of multi-dimensional obliquely reflected BSDEs: for $i \in \Lambda$ and $t \in [0,T]$,

$$\left\{\begin{array}{rcl} Y_i(t) & = & \xi_i + \displaystyle\int_t^T \psi(s, Y_i(s), Z_i(s), i)\, ds \\ & & - \displaystyle\int_t^T dK_i(s) + \int_t^T dL_i(s) - \int_t^T Z_i(s)\, dW(s), \\ Y_i(t) & \le & \displaystyle\min_{i' \neq i}\{Y_{i'}(t) + k(i,i')\}, \quad Y_i(t) \ge S(t), \\ & & \displaystyle\int_0^T \left(Y_i(s) - \min_{i' \neq i}\{Y_{i'}(s) + k(i,i')\}\right) dK_i(s) = 0, \\ & & \displaystyle\int_0^T (Y_i(t) - S(t))\, dL_i(t) = 0. \end{array}\right. \tag{1.7}$$

Here, S is a given $\{\mathcal{F}_t, 0 \le t \le T\}$-adapted process with some suitable regularity.

RBSDE (1.5) is more complicated than that of RBSDE (1.6) arising from the optimal switching problem for BSDEs. For each fixed $j \in \Pi$, if we do not impose the following constraint:

$$Y_{ij}(t) \ge \max_{j' \neq j}\{Y_{ij'}(t) - l(j,j')\}, \quad t \in [0,T], \tag{1.8}$$

and its related boundary condition:

$$\int_0^T \left(Y_{ij}(s) - \max_{j' \neq j}\{Y_{ij'}(s) - l(j,j')\}\right) dL_{ij}(s) = 0, \tag{1.9}$$

then we can take $L \equiv 0$, and RBSDE (1.5) is reduced to RBSDE (1.6).

RBSDE (1.5) evolves in the closure $\overline{Q}$ of domain Q:

$$\begin{array}{rcl} Q & := & \Big\{ (y_{ij}) \in \mathrm{R}^{m_1 \times m_2} : y_{ij} < y_{i'j} + k(i,i') \\ & & \quad \text{for any } i, i' \in \Lambda \text{ such that } i' \neq i \text{ and } j \in \Pi; \\ & & \qquad y_{ij} > y_{ij'} - l(j,j') \\ & & \quad \text{for any } j, j' \in \Pi \text{ such that } j' \neq j \text{ and } i \in \Lambda \Big\}, \end{array} \tag{1.10}$$

which is convex and unbounded. The boundary ∂Q of domain Q consists of the boundaries ∂D_{ij}^- and ∂D_{ij}^+, $(i,j) \in \Lambda \times \Pi$, with

$$D_{ij}^- := \{(y_{ij}) \in \mathrm{R}^{m_1 \times m_2} : y_{ij} < y_{i'j} + k(i,i'), \text{ for any } i' \in \Lambda \text{ such that } i' \neq i\}$$

and

$$D_{ij}^+ := \{(y_{ij}) \in \mathrm{R}^{m_1 \times m_2} : y_{ij} > y_{ij'} - l(j,j'), \text{ for any } j' \in \Pi \text{ such that } j' \neq j\}$$

for $(i,j) \in \Lambda \times \Pi$. That is,

$$\partial Q = \bigcup_{i=1}^{m_1} \bigcup_{j=1}^{m_2} \left(\partial D_{ij}^- \cup \partial D_{ij}^+ \right).$$

In the interior of $\overline{Q}$, each equation in (1.5) is independent of others. On the boundary, say ∂D_{ij}^- (resp. ∂D_{ij}^+), the (i,j)-th equation is switched to another one (i',j) (resp. (i,j')), and the solution is reflected along the oblique direction $-e_{ij}$ (resp. e_{ij}), which is the negative (resp. positive) direction of the (i,j)-th coordinate axis.

The existence of solution for RBSDE (1.5) constitutes a main contribution of this paper. We prove the existence by a penalization method. Proving the existence of solution of RBSDE (1.5) presents new difficulties when one follows our previous work [11] using the penalization method. In fact, in order to establish the a priori estimates which are essential for the proof of the existence, we have to use the representation of solutions to obliquely reflected BSDEs proved in [11], and we have to impose the additional technical condition that the generator ψ is uniformly bounded. The question of uniqueness is still an open problem.

The rest of the paper is organized as follows: in Section 2, we prove the existence of solution for RBSDE (1.5) by a penalization method. The last section is devoted to discussions on some possible extensions.

2 Existence of an adapted solution to the associated RBSDE

In this section, we state and prove our existence result for RBSDE (1.5).

We need the following additional technical assumption.

Hypothesis 2.1. *The generator ψ is uniformly bounded with respect to all its arguments.*

we shall use $\left|\psi\right|_\infty$ to denote the least upper bound of $|\psi|$.

Definition 2.1. *An adapted solution to RBSDE (1.5) is defined to be a set $(Y,Z,K,L) = \{Y(t),Z(t),K(t),L(t)\}_{t\in[0,T]}$ of predictable processes with values in $(R^{m_1\times m_2})^{1+d+1+1}$ such that P-a.s., $t \mapsto Y(t)$ is continuous, $t \mapsto K(t)$ and $t \mapsto L(t)$ are continuous and componentwisely increasing, $t \mapsto Z(t)$ belongs to $L^2(0,T;(R^{m_1\times m_2})^d)$, $t \mapsto \psi(t,Y_{ij}(t),Z_{ij}(t),i,j)$ belongs to $L^1(0,T;R^{m_1\times m_2})$ and P-a.s., RBSDE (1.5) holds for each $t \in [0,T]$.*

The main result of this paper is the following existence of an adapted solution to RBSDE (1.5).

Theorem 2.1. *Let Hypotheses 1.1, 1.2 and 2.1 be satisfied. Assume that*

$$\xi \in L^2(\Omega, \mathcal{F}_T, P; R^{m_1 \times m_2})$$

takes values in $\bar{Q}$. *Then RBSDE (1.5) has an adapted solution* (Y, Z, K, L) *in* $S^2 \times M^2 \times (N^2)^2$.

We first sketch the proof.

Sketch of the Proof: The proof is divided into five subsections. In Subsection 3.1, we introduce the penalized RBSDEs whose existence of solution follows from a slightly generalized result in [11]. In Subsection 3.2, we give the (implicit) representation of these solutions. In Subsection 3.3, we state a fundamental lemma and some (uniform) a priori estimates for these solutions. In Subsection 3.4, we prove the (monotone) convergence of these solutions. And the last subsection is devoted to checking out the boundary conditions.

2.1 The penalized RBSDEs

We shall use a penalization method to construct a solution to RBSDE (1.5). We observe (as mentioned in the introduction) that RBSDE (1.5) consists of the m_2 systems of m_1-dimensional obliquely reflected BSDEs of the form like (1.6):

$$\begin{cases} Y_{ij}(t) & = & \xi_{ij} + \int_t^T \psi(s, Y_{ij}(s), Z_{ij}(s), i, j)\, ds \\ & & - \int_t^T dK_{ij}(s) + \int_t^T dL_{ij}(s) - \int_t^T Z_{ij}(s)\, dW(s), \\ Y_{ij}(t) & \le & \min_{i' \neq i} \{Y_{i'j}(t) + k(i, i')\}, \\ & & \int_0^T \left(Y_{ij}(s) - \min_{i' \neq i} \{Y_{i'j}(s) + k(i, i')\} \right) dK_{ij}(s) = 0; \quad i \in \Lambda, \end{cases} \tag{2.1}$$

with the unknown processes being

$$(Y_{ij}, Z_{ij}, K_{ij}; i = 1, 2, \ldots, m_1)$$

(the process $(L_{1j}, \ldots, L_{m_1 j})$ is taken to be previously given) for $j = 1, 2, \ldots, m_2$. These m_2 systems have been well studied by Hu and Tang [11]. In RBSDE (1.5), they are coupled together by the processes $(L_{1j}, \ldots, L_{m_1 j})$ through the constraint

$$Y_{ij}(t) \ge \max_{j' \neq j} \{Y_{ij'}(t) - l(j, j')\}, \quad (i, j) \in \Lambda \times \Pi \tag{2.2}$$

and the minimal boundary condition:

$$\int_0^T \left(Y_{ij}(s) - \max_{j' \neq j} \{Y_{ij'}(s) - l(j, j')\} \right) dL_{ij}(s) = 0, \quad (i, j) \in \Lambda \times \Pi. \tag{2.3}$$

Therefore, it is natural to consider the following penalized system of RBSDEs (the unknown processes are $(Y_{ij}, Z_{ij}, K_{ij}; i \in \Lambda, j \in \Pi)$:

$$\left\{\begin{array}{rcl} Y_{ij}(t) & = & \xi_{ij} + \displaystyle\int_t^T \psi(s, Y_{ij}(s), Z_{ij}(s), i, j)\, ds \\ & & + n \displaystyle\sum_{j'=1}^{m_2} \int_t^T \left(Y_{ij}(s) - Y_{ij'}(s) + l(j,j')\right)^- \, ds \\ & & - \displaystyle\int_t^T dK_{ij}(s) - \int_t^T Z_{ij}(s)\, dW(s); \\ Y_{ij}(t) & \le & \min_{i' \neq i}\{Y_{i'j}(t) + k(i,i')\}; \\ & & \displaystyle\int_0^T \left(Y_{ij}(s) - \min_{i' \neq i}\{Y_{i'j}(s) + k(i,i')\}\right) dK_{ij}(s) = 0; \quad (i,j) \in \Lambda \times \Pi. \end{array}\right. \tag{2.4}$$

Note that when $j' = j$, we have, in view of Hypothesis 1.2 (ii),

$$(Y_{ij}(s) - Y_{ij'}(s) + l(j,j'))^- = 0. \tag{2.5}$$

Note also that for any integer n, the ij-th component of the generator of (2.4) depends also on $y_{ij'}, j' \neq j$. Hence we cannot apply directly the existence result in [11]. However, by slightly adapting the relevant arguments in [11], we have the following assertion.

Proposition 2.1. *For any integer n, RBSDE (2.4) has an adapted solution (Y^n, Z^n, K^n) in the space $S^2 \times M^2 \times N^2$.*

Proof. Let the integer n be fixed. For an integer m, consider the following penalized BSDE whose solution is denoted by $(Y^{n,m}, Z^{n,m})$:

$$\left\{\begin{array}{rcl} Y_{ij}(t) & = & \xi_{ij} + \displaystyle\int_t^T \psi(s, Y_{ij}(s), Z_{ij}(s), i, j)\, ds \\ & & + n \displaystyle\sum_{j'=1}^{m_2} \int_t^T \left(Y_{ij}(s) - Y_{ij'}(s) + l(j,j')\right)^- \, ds \\ & & - m \displaystyle\sum_{i'=1}^{m_1} \int_t^T \left(Y_{ij}(s) - Y_{i'j}(s) - k(i,i')\right)^+ \, ds \\ & & - \displaystyle\int_t^T Z_{ij}(s)\, dW(s); \quad (i,j) \in \Lambda \times \Pi. \end{array}\right. \tag{2.6}$$

From the comparison theorem for multi-dimensional BSDEs in [10], $\{Y_{ij}^{n,m}(t)\}_m$ is decreasing. Following the relevant arguments in [11], we prove that there exists an adapted solution (Y^n, Z^n, K^n) in the space $S^2 \times M^2 \times N^2$, and moreover

$$Y_{ij}^n(t) = \lim_{m \to \infty} Y_{ij}^{n,m}(t).$$

□

2.2 Representation and uniqueness of solution to the penalized RBSDE

Note again that for any integer n, the ij-th component of the generator of (2.4) depends also on $y_{ij'}, j' \neq j$. Hence we cannot apply directly the Representation Theorem 3.1 in [11].

Nevertheless, by defining a new generator

$$\widetilde{\psi}(r, y, z, i, j; n) := \psi(r, y, z, i, j) + n \sum_{j' \neq j} \left(y - Y^n_{ij'}(r) + l(j, j')\right)^- \tag{2.7}$$

for $(r, y, z) \in [t, T] \times R \times R^d$, and $(i, j) \in \Lambda \times \Pi$ to include $Y^n_{ij'}, j' \neq j$ in it, we have for each $j \in \Pi$, the triplet $(Y^n_{ij}, Z^n_{ij}, K^n_{ij}; i \in \Lambda)$ is an adapted solution of the following m_1-dimensional RBSDE:

$$\begin{cases} Y_{ij}(t) &= \xi_{ij} + \displaystyle\int_t^T \widetilde{\psi}(s, Y_{ij}(s), Z_{ij}(s), i, j; n)\, ds \\ & \quad - \displaystyle\int_t^T dK_{ij}(s) - \int_t^T Z_{ij}(s)\, dW(s); \\ Y_{ij}(t) &\leq \displaystyle\min_{i' \neq i} \{Y_{i'j}(t) + k(i, i')\}; \\ & \displaystyle\int_0^T \left(Y_{ij}(s) - \min_{i' \neq i}\{Y_{i'j}(s) + k(i, i')\}\right) dK_{ij}(s) = 0; \quad i \in \Lambda. \end{cases} \tag{2.8}$$

Then we can apply the Representation Theorem 3.1 in [11].

In order to state this representation theorem, first we introduce some notations.

Let $\{\theta_j\}_{j=0}^\infty$ be an increasing sequence of stopping times with values in $[t, T]$ and $\forall j$, α_j is an $\mathcal{F}_{\theta_j}$-measurable random variable with values in Λ, and χ is the indicator function. We define

$$a(s) := \alpha_0 \chi_{\{\theta_0\}}(s) + \sum_{j=1}^\infty \alpha_{j-1} \chi_{(\theta_{j-1}, \theta_j]}(s), \quad s \in [t, T].$$

The sequence $\{\theta_j, \alpha_j\}_{j=0}^\infty$ or $a(\cdot)$ is said to be an admissible switching strategy starting from the mode α_0, if there exists an integer-valued random variable N such that $\theta_N = T$, P-a.s. and $N \in L^2(\mathcal{F}_T)$.

We denote by $\mathcal{A}_t$ the set of all these admissible switching strategies and by $\mathcal{A}^i_t$ the subset of $\mathcal{A}$ consisting of admissible switching strategies starting from the mode i.

For any $a(\cdot) \in \mathcal{A}_t$, we define the associated (cost) process $A^{a(\cdot)}$ as follows:

$$A^{a(\cdot)}(s) = \sum_{j=1}^{N-1} k(\alpha_{j-1}, \alpha_j) \chi_{[\theta_j, T]}(s), \ s \in [t, T].$$

Obviously, $A^{a(\cdot)}(\cdot)$ is an adapted increasing càdlàg process, and $A^{a(\cdot)}(T) \in L^2(\mathcal{F}_T)$ thanks to the fact that $N \in L^2(\mathcal{F}_T)$.

Now we are in position to introduce the switched BSDE. For $(t,i,j)\in[0,T)\times\Lambda\times\Pi$ and $a(\cdot)\in\mathcal{A}_t^i$, $(U_j^{a(\cdot),n},V_j^{a(\cdot),n})$ is the unique solution to the following BSDE:

$$\begin{aligned} U_j(s) &= \xi_{a(T)j}+[A^{a(\cdot)}(T)-A^{a(\cdot)}(s)]+\int_s^T\widetilde{\psi}(r,U_j(r),V_j(r),a(r),j;n)\,dr \\ &\quad -\int_s^T V_j(r)\,dW(r),\quad s\in[t,T]. \end{aligned} \tag{2.9}$$

On the other hand, for $a(\cdot)\in\mathcal{A}_t^i$ of the following form:

$$a(s)=i\chi_t(s)+\sum_{p=1}^N\alpha_{p-1}\chi_{(\theta_{p-1},\theta_p]}(s),\quad s\in[t,T], \tag{2.10}$$

we define for $s\in[t,T]$ and $j\in\Pi$,

$$\begin{aligned} \widetilde{Y}_j^{a(\cdot),n}(s) &:= \sum_{p=1}^N Y^n_{\alpha_{p-1},j}(s)\chi_{[\theta_{p-1},\theta_p)}(s)+\xi_{a(T)}\chi_{\{T\}}(s), \\ \widetilde{Z}_j^{a(\cdot),n}(s) &:= \sum_{p=1}^N Z^n_{\alpha_{p-1},j}(s)\chi_{[\theta_{p-1},\theta_p)}(s), \\ \widetilde{K}_j^{a(\cdot),n}(s) &:= \sum_{p=1}^N\int_{\theta_{p-1}\wedge s}^{\theta_p\wedge s}dK^n_{\alpha_{p-1},j}(r), \end{aligned} \tag{2.11}$$

and

$$\widetilde{A}_j^{a(\cdot),n}(s)=\sum_{p=1}^{N-1}[Y^n_{\alpha_p,j}(\theta_p)+k(\alpha_{p-1},\alpha_p)-Y^n_{\alpha_{p-1},j}(\theta_p)]\chi_{[\theta_p,T]}(s),\quad s\in[t,T]. \tag{2.12}$$

$\widetilde{A}_j^{a(\cdot),n}$ is an increasing process due to the fact that $Y^n(s)$ satisfies the boundary condition in (2.4), $\forall s\in[t,T]$. Then, for each $j\in\Pi$, the triplet $(\widetilde{Y}_j^{a(\cdot),n},\widetilde{Z}_j^{a(\cdot),n},\widetilde{K}_j^{a(\cdot),n})$ is a solution to the following BSDE:

$$\begin{aligned} &\widetilde{Y}_j^{a(\cdot),n}(s) \\ =\ &\xi_{a(T)j}-[(\widetilde{K}_j^{a(\cdot),n}(T)+\widetilde{A}_j^{a(\cdot),n}(T))-(\widetilde{K}_j^{a(\cdot),n}(s)+\widetilde{A}_j^{a(\cdot),n}(s))]+A^{a(\cdot)}(T)-A^{a(\cdot)}(s) \\ &+\int_s^T\widetilde{\psi}(r,\widetilde{Y}_j^{a(\cdot),n}(r),\widetilde{Z}_j^{a(\cdot),n}(r),a(r),j;n)\,dr-\int_s^T\widetilde{Z}_j^{a(\cdot),n}(r)\,dW(r). \end{aligned} \tag{2.13}$$

As

$$Y^n_{a(s)j'}(s)=\widetilde{Y}_{j'}^{a(\cdot),n}(s),\quad a.e.\quad s\in[t,T],$$

BSDE (2.9) can be rewritten as the following equation:

$$\begin{aligned} U_j(s) &= \xi_{a(T)j}+[A^{a(\cdot)}(T)-A^{a(\cdot)}(s)]+\int_s^T\psi(r,U_j(r),V_j(r),a(r),j)\,dr \\ &\quad +n\sum_{j'\neq j}\int_s^T(U_j(r)-\widetilde{Y}_{j'}^{a(\cdot),n}(r)+l(j,j'))^-\,dr-\int_s^T V_j(r)\,dW(r), \\ &\qquad s\in[t,T]. \end{aligned} \tag{2.14}$$

We are now ready to state the representation formula which is taken from Theorem 3.1 in [11].

Proposition 2.2. *Assume that $a(\cdot) \in \mathcal{A}_t^i$. Then we have*

$$\operatorname*{ess\,sup}_{a(\cdot)\in\mathcal{A}_t^i} \left(\widetilde{Y}_j^{a(\cdot),n}(s) - U_j^{a(\cdot),n}(s)\right) = 0, \quad s \in [t,T], \quad j \in \Pi, \tag{2.15}$$

which, putting in particular $s = t$, implies

$$Y_{ij}^n(t) = \operatorname*{essinf}_{a(\cdot)\in\mathcal{A}_t^i} U_j^{a(\cdot),n}(t), \quad j \in \Pi.$$

It is crucial to observe that the above representation formula is implicit since $U_j^{a(\cdot),n}$ still depends on Y^n. However, it is sufficient for us to deduce the a priori estimates. Also, it is sufficient for us to deduce the uniqueness of the solution to the penalized RBSDE (2.4). In fact, if we have two solutions $(Y_{ij}^{k,n}, Z_{ij}^{k,n}, K_{ij}^{k,n}; i \in \Lambda)$ with $k = 1, 2$, we can define $(U_j^{k,a(\cdot),n}, V_j^{k,a(\cdot),n}), k = 1, 2$, for $(t, i, j) \in [0, T) \times \Lambda \times \Pi$ and $a(\cdot) \in \mathcal{A}_t^i$, as the unique solution to BSDE (2.9) with Y_{ij}^n being replaced with $Y_{ij}^{1,n}$ and $Y_{ij}^{2,n}$, respectively. We have the estimate for $U_j^{1,a(\cdot),n} - U_j^{2,a(\cdot),n}$:

$$|U_j^{1,a(\cdot),n}(t) - U_j^{2,a(\cdot),n}(t)|^2 \leq C_n E[\int_t^T |Y^{1,n}(s) - Y^{2,n}(s)|^2 ds|F_t]$$

for some constant C_n. Then applying the above representation and Gronwall inequality, we obtain that $Y^{1,n} = Y^{2,n}$, which gives the uniqueness.

2.3 A basic lemma and a priori estimates

For each integer n, let $(Y^n, Z^n, K^n) \in S^2 \times M^2 \times N^2$ be the adapted solution of RBSDE (2.4). Intuitively, as n tends to $+\infty$, we expect that the sequence of solutions

$$\{(Y^n, Z^n, K^n)\}_{n=1}^{\infty}$$

together with the penalty term

$$L_{ij}^n(t) := n \sum_{j'=1}^{m_2} \int_0^t \left(Y_{ij}(s) - Y_{ij'}(s) + l(j, j')\right)^- ds, \quad (t, i, j) \in [0, T] \times \Lambda \times \Pi$$

will have a limit (Y, Z, K, L), which solves RBSDE (1.5).

For this purpose, it is crucial to prove that the penalty term is bounded in some suitable sense. Then we are naturally led to compute

$$\left(Y_{ij}(t) - Y_{ij'}(t) + l(j, j')\right)^-,$$

using Itô-Meyer's formula, as done in [11]. However, in our present situation, the additional term K^n appears in RBSDE (2.4), which gives rise to a serious difficulty to derive the bound of L^n in the preceding procedure. In what follows, we shall use the representation result for Y^n of Proposition 2.2 to get around the difficulty.

We have the following basic lemma.

Lemma 2.1. *For $j, j' \in \Pi$ and $a(\cdot) \in \mathcal{A}_t^i$, we have*

$$n\left(U_j^{a(\cdot),n}(s) - \widetilde{Y}_{j'}^{a(\cdot),n}(s) + l(j,j')\right)^- \leq 2\Big|\psi\Big|_\infty, \quad s \in [t,T]. \tag{2.16}$$

Here, $U_j^{a(\cdot),n}$ and $\widetilde{Y}_{j'}^{a(\cdot),n}$ are defined by (2.9) and (2.11), respectively.

Proof. We suppress the superscripts $(a(\cdot), n)$ of $U_j^{a(\cdot),n}, U_{j'}^{a(\cdot),n}, V_j^{a(\cdot),n}, V_{j'}^{a(\cdot),n}$, $\widetilde{Y}_j^{a(\cdot),n}$, $\widetilde{Y}_{j'}^{a(\cdot),n}$, $\widetilde{Y}_{j''}^{a(\cdot),n}$ and $\widetilde{Z}_{j'}^{a(\cdot),n}$ for simplicity. The whole proof consists of the following two steps.

Step 1. Calculation of the process $\overline{U}_{jj'}(s)^-$, where $\overline{U}_{jj'}(s) := U_j(s) - \widetilde{Y}_{j'}(s) + l(j,j'), s \in [t,T]$ using Itô-Meyer's formula.

In view of (2.14) and (2.13), the process $\overline{U}_{jj'}(s), s \in [t,T]$ satisfies the following BSDE:

$$\begin{aligned}
& \overline{U}_{jj'}(s) \\
= \; & \overline{U}_{jj'}(T) + \int_s^T \Big[\psi(r, U_j(r), V_j(r), a(r), j) - \psi(r, \widetilde{Y}_{j'}(r), \widetilde{Z}_{j'}(r), a(r), j')\Big]\, dr \\
& + n \sum_{j'' \neq j} \int_s^T (U_j - \widetilde{Y}_{j''} + l(j, j''))(r)^-\, dr - n \sum_{j'' \neq j'} \int_s^T (\widetilde{Y}_{j'} - \widetilde{Y}_{j''} + l(j', j''))(r)^-\, dr \\
& + \int_s^T d\left(\widetilde{K}_j^{a(\cdot),n}(r) + \widetilde{A}_j^{a(\cdot),n}(r)\right) - \int_s^T (V_j(r) - \widetilde{Z}_{j'}(r))\, dW(r), \quad s \in [t,T].
\end{aligned} \tag{2.17}$$

Applying Itô-Meyer's formula (see, e.g. Meyer [13]), we have

$$\begin{aligned}
& \overline{U}_{jj'}(s)^- + n \sum_{j'' \neq j} \int_s^T \chi_{\mathcal{L}_{jj'}^-}(r)(U_j - \widetilde{Y}_{j''} + l(j, j''))(r)^-\, dr \\
& - n \sum_{j'' \neq j'} \int_s^T \chi_{\mathcal{L}_{jj'}^-}(r)(\widetilde{Y}_{j'} - \widetilde{Y}_{j''} + l(j', j''))(r)^-\, dr + \int_s^T d\widehat{L}_{jj'}(r) \\
= \; & - \int_s^T \chi_{\mathcal{L}_{jj'}^-}(r) \Big[\psi(r, U_j(r), V_j(r), a(r), j) - \psi(r, \widetilde{Y}_{j'}(r), \widetilde{Z}_{j'}(r), a(r), j')\Big]\, dr \\
& - \int_s^T \chi_{\mathcal{L}_{jj'}^-}(r-)\, d\left(\widetilde{K}_j^{a(\cdot),n}(r) + \widetilde{A}_j^{a(\cdot),n}(r)\right) \\
& + \int_s^T \chi_{\mathcal{L}_{jj'}^-}(r-)(V_j(r) - \widetilde{Z}_{j'}(r))\, dW(r), \quad s \in [t,T]
\end{aligned} \tag{2.18}$$

where

$$\mathcal{L}_{jj'}^- := \{(s,\omega) \in [t,T] \times \Omega : \overline{U}_{jj'}(s) < 0\}, \tag{2.19}$$

and $\widehat{L}_{jj'}$ is a càdlàg increasing process. The above equation can be rewritten as

$$\begin{aligned}
&\overline{U}_{jj'}(s)^- + n\int_s^T \chi_{\mathcal{L}^-_{jj'}}(U_j - \widetilde{Y}_{j'} + l(j,j^{'}))(r)^-\,dr + \int_s^T d\widehat{L}_{jj'}(r)\\
= & -\int_s^T \chi_{\mathcal{L}^-_{jj'}}(r)\left[\psi(r,U_j(r),V_j(r),a(r),j) - \psi(r,\widetilde{Y}_{j'}(r),\widetilde{Z}_{j'}(r),a(r),j')\right]dr\\
& -\int_s^T \chi_{\mathcal{L}^-_{jj'}}(r-)\,d(\widetilde{K}_j^{a(\cdot),n}(r) + \widetilde{A}_j^{a(\cdot),n}(r))\\
& +\int_s^T \chi_{\mathcal{L}^-_{jj'}}(r-)\left(V_j(r) - \widetilde{Z}_{j'}(r)\right)dW(r)\\
& +n\int_s^T I_{1,jj'}(r)\,dr + n\sum_{j''\neq j, j''\neq j'}\int_s^T I_{2,jj'j''}(r)\,dr
\end{aligned} \tag{2.20}$$

where the two integrands $I_{1,jj'}$ and $I_{2,jj'j''}$ are defined as follows

$$I_{1,jj'}(r) := \chi_{\mathcal{L}^-_{jj'}}(r)(\widetilde{Y}_{j'} - \widetilde{Y}_j + l(j',j))^-, \quad r\in[t,T] \tag{2.21}$$

and

$$I_{2,jj'j''}(r) := \chi_{\mathcal{L}^-_{jj'}}(r)[(\widetilde{Y}_{j'} - \widetilde{Y}_{j''} + l(j',j^{''}))(r)^- - (U_j - \widetilde{Y}_{j''} + l(j,j''))(r)^-], \quad r\in[t,T]. \tag{2.22}$$

Step 2. The BSDE for the process $\{(U_j(s) - \widetilde{Y}_{j'}(s) + l(j,j'))^-, s\in[t,T]\}$**.**

In view of Proposition 2.2,

$$\widetilde{Y}_j \le U_j,$$

we have

$$I_{1,jj'} \le \chi_{\mathcal{L}^-_{jj'}}(\widetilde{Y}_{j'} - U_j + l(j',j))^- = 0, \quad j,j'\in\Pi, \tag{2.23}$$

thanks to the fact that

$$l(j,j') + l(j',j) > l(j,j) = 0.$$

Hence,

$$I_{1,jj'}(r) = 0.$$

Now we can rewrite (2.20) as the following equation:

$$\begin{aligned}
&(U_j(s) - \widetilde{Y}_{j'}(s) + l(j,j'))^-\\
= \;& n\sum_{j''\neq j, j''\neq j'}\int_s^T I_{2,jj'j''}(r)\,dr + \int_s^T I_{3,jj'}(r)\,dr\\
& -\int_s^T d\widehat{L}_{jj'}(r) - \int_s^T \chi_{\mathcal{L}^-_{jj'}}(r-)\,d(\widetilde{K}_j^{a(\cdot),n}(r) + \widetilde{A}_j^{a(\cdot),n}(r))\\
& -n\int_s^T (U_j(r) - \widetilde{Y}_{j'}(r) + l(j,j^{'}))^-\,dr\\
& +\int_s^T \chi_{\mathcal{L}^-_{jj'}}(r-)\left(V_j(r) - \widetilde{Z}_{j'}(r)\right)dW(r),
\end{aligned} \tag{2.24}$$

where

$$I_{3,jj'}(r) := \chi_{\mathcal{L}^-_{jj'}}(r)\left[\psi(r, \widetilde{Y}_{j'}(r), \widetilde{Z}_{j'}(r), a(r), j') - \psi(r, U_j(r), V_j(r), a(r), j)\right] \tag{2.25}$$

for $r \in [t, T]$.

We now show that

$$I_{2,jj'j''} \leq 0. \tag{2.26}$$

In fact, for $j, j', j'' \in \Pi$, taking into consideration the elementary inequality that $x_1^- - x_2^- \leq (x_1 - x_2)^-$, for any two real numbers x_1 and x_2, we have

$$\begin{aligned} & I_{2,jj'j''} = \chi_{\mathcal{L}^-_{jj'}}[(\widetilde{Y}_{j'} - \widetilde{Y}_{j''} + l(j', j''))(r)^- - (U_j - \widetilde{Y}_{j''} + l(j, j''))(r)^-] \\ \leq \; & \chi_{\mathcal{L}^-_{jj'}}\left(\widetilde{Y}_{j'} - U_j + l(j', j'') - l(j, j'')\right)^- = 0. \end{aligned} \tag{2.27}$$

The last equality holds in the last relations, since

$$\{y \in R^m : y_j - y_{j'} + l(j, j') < 0\} \cap \{y \in R^m : y_{j'} - y_j + l(j', j'') - l(j, j'') < 0\} = \emptyset,$$

thanks to Hypothesis 1.2 (iv), i.e.,

$$l(j, j') + l(j', j'') > l(j, j'').$$

Define for $r \in [t, T]$,

$$\begin{aligned} I_{jj'}(r) \;\; := \;\; & n \sum_{j'' \neq j, j'' \neq j'} \int_t^r I_{2,jj'j''}(s)\, ds \\ & - \int_t^r d\widehat{L}_{jj'}(s) - \int_t^r \chi_{\mathcal{L}^-_{jj'}}(s-)\, d(\widetilde{K}_j^{a(\cdot),n}(s) + \widetilde{A}_j^{a(\cdot),n}(s)). \end{aligned}$$

Obviously, the process $I_{jj'}(\cdot)$ is a càdlàg decreasing process for any $(j, j') \in \Pi \times \Pi$. BSDE (2.24) is finally written as the following equation:

$$\begin{aligned} & (U_j(s) - \widetilde{Y}_{j'}(s) + l(j, j'))^- \\ = \; & \int_t^T dI_{jj'}(r) + \int_s^T I_{3,jj'}(r)\, dr - n \int_s^T (U_j(r) - \widetilde{Y}_{j'}(r) + l(j, j'))^-\, dr \\ & + \int_s^T \chi_{\mathcal{L}^-_{jj'}}(r-)\left(V_j(r) - \widetilde{Z}_{j'}(r)\right) dW(r), \quad s \in [t, T]. \end{aligned} \tag{2.28}$$

Then, we have the following formula

$$\begin{aligned} (U_j(s) - \widetilde{Y}_{j'}(s) + l(j, j'))^- \;\; = \;\; & E\left[\int_s^T I_{3,jj'}(r) \exp\left[-n(r - s)\right] dr \,\middle|\, \mathcal{F}_s\right] \\ & + E\left[\int_s^T \exp\left[-n(r - s)\right] dI_{jj'}(r) \,\middle|\, \mathcal{F}_s\right] \\ \leq \;\; & 2\Big|\psi\Big|_\infty \int_s^T \exp\left[-n(r - s)\right] dr. \end{aligned} \tag{2.29}$$

Therefore, we have

$$n\left(U_j^{a(\cdot),n}(s) - \widetilde{Y}_{j'}^{a(\cdot),n}(s) + l(j, j')\right)^- \leq 2\Big|\psi\Big|_\infty, \quad s \in [t, T].$$

This ends the proof. □

Thanks to this basic lemma, we deduce easily the following a priori estimates.

Proposition 2.3. *(i) The sequence* $\{Y_{ij}^n(t)\}_{n=1}^\infty$ *is increasing. Moreover,*

$$-E\left[|\xi||\mathcal{F}_t\right] - |\psi|_\infty T \le Y_{ij}^n(t) \le E\left[|\xi||\mathcal{F}_t\right] + 3|\psi|_\infty T; \quad E\left[\sup_t |Y_{ij}^n(t)|^2\right] \le C, \quad (2.30)$$

where $C > 0$ *is a constant.*

(ii) We have

$$n\left(Y_{ij}^n(t) - Y_{ij'}^n(t) + l(j,j')\right)^- \le 2\Big|\psi\Big|_\infty. \quad (2.31)$$

Proof. (i) According to the comparison theorem for multi-dimensional BSDEs in [10],

$$Y_{ij}^{n,m}(t) \le Y_{ij}^{n+1,m}(t),$$

where $Y_{ij}^{n,m}$ is defined via (2.6). Hence, the sequence $\{Y_{ij}^n(t)\}_{n=1}^\infty$ is increasing by taking the limit when m tends to ∞.

We have the following two facts in view of (2.14):

(1) $U_j^{a(\cdot),n}(s) \ge -E\left[|\xi||\mathcal{F}_s\right] - |\psi|_\infty T$.

(2) Taking $\bar{a}(\cdot) \equiv i$, we have, from Lemma 2.1,

$$U_j^{\bar{a}(\cdot),n}(s) \le E\left[|\xi||\mathcal{F}_s\right] + |\psi|_\infty T + 2|\psi|_\infty T. \quad (2.32)$$

In view of the representation formula in Proposition 2.2, we conclude the proof.

(ii) Putting $s = t$ in (2.16), we obtain

$$n\left(U_j^{a(\cdot),n}(t) - Y_{ij'}^n(t) + l(j,j')\right) \ge -2\Big|\psi\Big|_\infty.$$

From Proposition 2.2, we deduce that

$$n\left(Y_{ij}^n(t) - Y_{ij'}^n(t) + l(j,j')\right) \ge -2\Big|\psi\Big|_\infty,$$

and the proof is complete. □

2.4 Convergence of solutions

We first prove that (Z^n, K^n) is bounded.

Lemma 2.2. *The pair of processes* (Z_{ij}^n, K_{ij}^n) *are uniformly bounded in* $M^2 \times N^2$ *for* $(i,j) \in \Lambda \times \Pi$.

Proof. From the RBSDE for Y_{ij}^n, in view of Hypothesis 2.1, using Itô's formula and Proposition 2.3, we have

$$
\begin{aligned}
& E|Y_{ij}^n(0)|^2 + E\int_0^T |Z_{ij}^n(s)|^2\,ds \\
\le\ & E|\xi_{ij}|^2 + 2\sum_{j'=1}^{m_2} E\int_0^T |Y_{ij}^n(s)| n\left(Y_{ij}^n(s) - Y_{ij'}^n(s) + l(j,j')\right)^-\,ds \\
& +2E\int_0^T |Y_{ij}^n(s)|\cdot|\psi(s, Y_{ij}^n(s), Z_{ij}^n(s), i, j)|\,ds \\
& +2E\int_0^T |Y_{ij}^n(s)|\,dK_{ij}^n(s) \\
\le\ & C + CE\int_0^T |Y_{ij}^n(s)|\,ds + 2E\left[\sup_t |Y_{ij}^n(t)| K_{ij}^n(T)\right] \\
\le\ & C_\epsilon + \epsilon E[(K_{ij}^n(T))^2]
\end{aligned}
\tag{2.33}
$$

and

$$
\begin{aligned}
& E[(K_{ij}^n(T))^2] \\
\le\ & CE|\xi_{ij}|^2 + CE|Y_{ij}^n(0)|^2 \\
& +C\sum_{j'=1}^{m_2} E\int_0^T \left[n\left(Y_{ij}^n(s) - Y_{ij'}^n(s) + l(j,j')\right)^-\right]^2\,ds \\
& +CE\int_0^T |\psi(s, Y_{ij}^n(s), Z_{ij}^n(s), i, j)|^2\,ds + CE\int_0^T |Z_{ij}^n(s)|^2\,ds \\
\le\ & C + CE\int_0^T |Z_{ij}^n(s)|^2\,ds.
\end{aligned}
\tag{2.34}
$$

Combining the above two inequalities by taking a sufficiently small $\epsilon > 0$, we conclude the proof. $\square$

Define

$$\beta_{ij}^n(s) := n\sum_{j'=1}^{m_2} \left(Y_{ij}^n(s) - Y_{ij'}^n(s) + l(j,j')\right)^-. \tag{2.35}$$

Then

$$L_{ij}^n(t) = \int_0^t \beta_{ij}^n(s)\,ds. \tag{2.36}$$

Next, we prove that K^n is absolutely continuous whose derivative is uniformly bounded.

Lemma 2.3. *For $(i,j) \in \Lambda \times \Pi$ and an integer n, there is a uniformly bounded process α_{ij}^n such that K_{ij}^n has the following form:*

$$K_{ij}^n(t) = \int_0^t \alpha_{ij}^n(s)\,ds, \quad t \in [0,T]. \tag{2.37}$$

Proof. Fix the integer n. Consider the following penalized BSDEs:

$$\begin{aligned} Y_{ij}(t) &= \xi_{ij} + \int_t^T \left[\psi(s, Y_{ij}^n(s), Z_{ij}^n(s), i, j) + \beta_{ij}^n(s)\right] ds \\ &\quad -m \sum_{i'=1}^{m_1} \int_t^T \left(Y_{ij} - Y_{i'j} - k(i,i')\right)^+ ds \\ &\quad - \int_t^T Z_{ij}(s)\, dW(s), \end{aligned} \tag{2.38}$$

with $(i,j) \in \Lambda \times \Pi$ and $t \in [0,T]$. It has a unique solution, denoted by $(\bar{Y}_{ij}^{n,m}, \bar{Z}_{ij}^{n,m})$.

Proceeding similarly (in fact, much more simpler) as in Lemma 2.1, we can prove that for a constant $C > 0$,

$$\alpha_{ij}^{n,m} := m \sum_{i'=1}^{m_1} \left(\bar{Y}_{ij}^{n,m} - \bar{Y}_{i'j}^{n,m} - k(i,i')\right)^+ \leq C. \tag{2.39}$$

Therefore, $\{\alpha_{ij}^{n,m}\}_{m=1}^{\infty}$ has a weak limit in M^2, denoted by α_{ij}^n. Then α_{ij}^n is also uniformly bounded by the same constant C.

Define

$$\bar{K}_{ij}^{n,m}(t) := \int_0^t \alpha_{ij}^{n,m}(s)\, ds, \quad t \in [0,T]. \tag{2.40}$$

From [11], we have

$$\begin{aligned} \lim_{m\to\infty} \bar{Y}_{ij}^{n,m}(t) &= Y_{ij}^n(t), \\ \lim_{m\to\infty} \bar{Z}_{ij}^{n,m}(t) &= Z_{ij}^n(t), \\ \lim_{m\to\infty} \bar{K}_{ij}^{n,m}(t) &= \int_0^t \alpha_{ij}^n(s)\, ds = K_{ij}^n(t). \end{aligned} \tag{2.41}$$

□

Now we are able to state the convergence result.

Lemma 2.4. *The sequence $\{Y^n, Z^n\}$ has a strong limit (Y, Z) in $S^2 \times M^2$. The two sequences $\{\alpha^n\}$ and $\{\beta^n\}$ have subsequences which converge to α and β weakly in M^2, respectively.*

Proof. Note that Y_{ij}^n is increasing in n. In view of Proposition 2.3 and applying the dominated convergence theorem, we deduce easily the strong convergence of $\{Y^n\}$ in the space M^2. Note that (Y^n, Z^n) solves the following BSDE:

$$\begin{aligned} Y_{ij}^n(t) &= \xi_{ij} + \int_t^T \left[\psi(s, Y_{ij}^n(s), Z_{ij}^n(s), i, j) + \beta_{ij}^n(s) - \alpha_{ij}^n(s)\right] ds \\ &\quad - \int_t^T Z_{ij}^n(s)\, dW(s), \quad t \in [0,T], \quad (i,j) \in \Lambda \times \Pi, \end{aligned} \tag{2.42}$$

with $\{\alpha^n\}$ and $\{\beta^n\}$ being uniformly bounded.

We now prove the strong convergence of Z^n. Using Itô's formula, we have

$$
\begin{aligned}
& |Y_{ij}^n(0) - Y_{ij}^m(0)|^2 + E\int_0^T |Z_{ij}^n(s) - Z_{ij}^m(s)|^2\, ds \\
= \; & 2E\int_0^T \left(Y_{ij}^n(s) - Y_{ij}^m(s)\right)\left(\psi(s, Y_{ij}^n(s), Z_{ij}^n(s), i, j) - \alpha_{ij}^n(s) + \beta_{ij}^n(s)\right)\, ds \\
& -2E\int_0^T \left(Y_{ij}^n(s) - Y_{ij}^m(s)\right)\left(\psi(s, Y_{ij}^m(s), Z_{ij}^m(s), i, j) - \alpha_{ij}^m(s) + \beta_{ij}^m(s)\right)\, ds,
\end{aligned}
\tag{2.43}
$$

from which we deduce immediately that

$$
\lim_{n,m\to\infty} E\int_0^T |Z^n(s) - Z^m(s)|^2\, ds = 0. \tag{2.44}
$$

It is routine to show the strong convergence of $\{Y^n\}$ in the space S^2.

Since $\{\alpha^n\}$ and $\{\beta^n\}$ are uniformly bounded, the last assertion of the lemma is obvious. □

Define for $(i, j, t) \in \Lambda \times \Pi \times [0, T]$,

$$
K_{ij}(t) := \int_0^t \alpha_{ij}(s)\, ds, \quad L_{ij}(t) := \int_0^t \beta_{ij}(s)\, ds \tag{2.45}
$$

and

$$
K := (K_{ij}), \quad L := (L_{ij}). \tag{2.46}
$$

Finally, we shall show that (Y, Z, K, L) solves RBSDE (1.5).

In fact, it suffices to take the weak limit in $L^2(\mathcal{F}_T)$ in BSDE (2.42) along a suitable subsequence to deduce that (Y, Z, K, L) solves the following BSDE:

$$
\begin{aligned}
Y_{ij}(t) \;=\; & \xi_{ij} + \int_t^T \psi(s, Y_{ij}(s), Z_{ij}(s), i, j)\, ds - \int_t^T dK_{ij}(s) \\
& + \int_t^T dL_{ij}(s) - \int_t^T Z_{ij}(s)\, dW(s), \quad (i, j) \in \Lambda \times \Pi.
\end{aligned}
\tag{2.47}
$$

It remains to check out the boundary conditions, which will be given in the next subsection.

2.5 Boundary conditions

Let us first prove that $Y(t) \in \bar{Q}$.

On the one hand, as (Y^n, Z^n, K^n) satisfies (2.4), we have

$$
Y_{ij}^n(t) \le \min_{i' \ne i}\{Y_{i'j}^n(t) + k(i, i')\},
$$

from which we deduce, by taking limit when n tends to ∞, that

$$
Y_{ij}(t) \le \min_{i' \ne i}\{Y_{i'j}(t) + k(i, i')\}. \tag{2.48}
$$

On the other hand, from Proposition 2.3,

$$\left(Y^n_{ij}(t) - Y^n_{ij'}(t) + l(j,j')\right)^- \le \frac{2\left|\psi\right|_\infty}{n}.$$

Sending n to ∞, we deduce that

$$\left(Y_{ij}(t) - Y_{ij'}(t) + l(j,j')\right)^- = 0. \tag{2.49}$$

(2.48) and (2.49) shows that $Y(t) \in \bar{Q}$.

Now we check out the minimal boundary conditions.

From (2.4), we have

$$E\int_0^T \left(Y^n_{ij}(s) - \min_{i'\neq i}\{Y^n_{i'j}(s) + k(i,i')\}\right)^- \alpha^n_{ij}(s)\,ds = 0. \tag{2.50}$$

Setting $n \to \infty$, we have

$$E\int_0^T \left(Y_{ij}(s) - \min_{i'\neq i}\{Y_{i'j}(s) + k(i,i')\}\right)^- \alpha_{ij}(s)\,ds = 0. \tag{2.51}$$

On the other hand, from the construction, we have

$$E\int_0^T \left(Y^n_{ij}(s) - \max_{j'\neq j}\{Y^n_{ij'}(s) - l(j,j')\}\right)^+ \beta^n_{ij}(s)\,ds = 0. \tag{2.52}$$

Setting $n \to \infty$, we have

$$E\int_0^T \left(Y_{ij}(s) - \max_{j'\neq j}\{Y_{ij'}(s) - l(j,j')\}\right)^+ \beta_{ij}(s)\,ds = 0. \tag{2.53}$$

The proof is then complete.

3 Concluding remarks

In this paper, we proved the existence of solution to RBSDE (1.5). But the question of uniqueness remains open.

On the other hand, there exist different methods in the literature for the study of switching control and game problems. For the classical method of quasi-variational inequalities, the reader is referred to the book of Bensoussan and Lions [1]. See Tang and Yong [18], Pham, Ly Vath and Zhou [15] and Tang and Hou [17] and the references therein for the theory of variational inequalities and the dynamic programming for optimal stochastic switching control and switching games. But these works are restricted to the Markovian case. Recently, using the method of Snell envelope (see, e.g. El Karoui [6]) combined with the theory of scalar valued RBSDEs, Hamadène and Jeanblanc [9] studied the switching problem in the non-Markovian context. The obliquely reflected BSDE approach, first fully developed in Hu and Tang [11] for optimal stochastic switching and taking the advantage of the theory and techniques of BSDEs, permits to state and solve these problems in a rather general non-Markovian framework. The link between the solution of RBSDE (1.5) and the problem of switching games constitutes another challenge.

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